

Extending Baire-one functions on compact spaces

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Abstract

We answer a question of O. Kalenda and J. Spurný from [8] and give an example of a completely regular hereditarily Baire space X and a Baire-one function $f : X \rightarrow [0, 1]$ which can not be extended to a Baire-one function on βX .

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1. Introduction

The classical Kuratowski's extension theorem [14, 35.VI] states that any map $f : E \rightarrow Y$ of the first Borel class to a Polish space Y can be extended to a map $g : X \rightarrow Y$ of the first Borel class if E is a G_δ -subspace of a metrizable space X . Non-separable version of Kuratowski's theorem was proved by Hansell [4, Theorem 9], while abstract topological versions of Kuratowski's theorem were developed in [5, 8, 9].

Recall that a map $f : X \rightarrow Y$ between topological spaces X and Y is said to be

- *Baire-one*, $f \in B_1(X, Y)$, if it is a pointwise limit of a sequence of continuous maps $f_n : X \rightarrow Y$;
- *functionally F_σ -measurable* or *of the first functional Borel class*, $f \in K_1(X, Y)$, if the preimage $f^{-1}(V)$ of any open set $V \subseteq Y$ is a union of a sequence of zero sets in X .

Notice that every functionally F_σ -measurable map belongs to the first Borel class for any X and Y ; the converse inclusion is true for perfectly normal X ; moreover, for a topological space X and a metrizable separable connected and locally path-connected space Y we have the equality $B_1(X, Y) = K_1(X, Y)$ (see [10]).

Kalenda and Spurný proved the following result [8, Theorem 13].

Theorem A. *Let E be a Lindelöf hereditarily Baire subset of a completely regular space X and $f : E \rightarrow \mathbb{R}$ be a Baire-one function. Then there exists a Baire-one function $g : X \rightarrow \mathbb{R}$ such that $g = f$ on E .*

The simple example shows that the assumption that E is hereditarily Baire cannot be omitted: if A and B are disjoint dense subsets of $E = \mathbb{Q} \cap [0, 1]$ such that $E = A \cup B$ and $X = [0, 1]$ or $X = \beta E$, then the characteristic function $f = \chi_A : E \rightarrow \mathbb{R}$ can not be extended to a Baire-one function on X . In connection with this the following question was formulated in [8, Question 1].

Question 1. *Let X be a hereditarily Baire completely regular space and f a Baire-one function on X . Can f be extended to a Baire-one function on βX ?*

We answer the question of Kalenda and Spurný in negative. We introduce a notion of functionally countably fragmented map (see definitions in Section 2) and prove that for a Baire-one function $f : X \rightarrow \mathbb{R}$ on a completely regular space X the following conditions are equivalent: (i) f is functionally countably fragmented; (ii) f can be extended to a Baire-one function on βX . In Section 3 we give an example of a completely regular hereditarily Baire (even scattered) space X and a Baire-one function $f : X \rightarrow [0, 1]$ which is not functionally countably fragmented and consequently can not be extended to a Baire-one function on βX .

2. Extension of countably fragmented functions

Let X be a topological space and (Y, d) be a metric space. A map $f : X \rightarrow Y$ is called ε -fragmented for some $\varepsilon > 0$ if for every closed nonempty set $F \subseteq X$ there exists a nonempty relatively open set $U \subseteq F$ such that $\text{diam} f(U) < \varepsilon$. If f is ε -fragmented for every $\varepsilon > 0$, then it is called *fragmented*.

Let $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ be a transfinite sequence of subsets of a topological space X . Following [6], we define \mathcal{U} to be *regular* in X , if

- (a) each U_ξ is open in X ;
- (b) $\emptyset = U_0 \subset U_1 \subset U_2 \subset \dots \subset U_\alpha = X$;
- (c) $U_\gamma = \bigcup_{\xi < \gamma} U_\xi$ for every limit ordinal $\gamma \in [0, \alpha]$.

Proposition 1. *Let X be a topological space, (Y, d) be a metric space and $\varepsilon > 0$. For a map $f : X \rightarrow Y$ the following conditions are equivalent:*

- (1) f is ε -fragmented;
- (2) there exists a regular sequence $\mathcal{U} = (U_\xi : \xi \in [0, \alpha])$ in X such that $\text{diam}f(U_{\xi+1} \setminus U_\xi) < \varepsilon$ for all $\xi \in [0, \alpha]$.

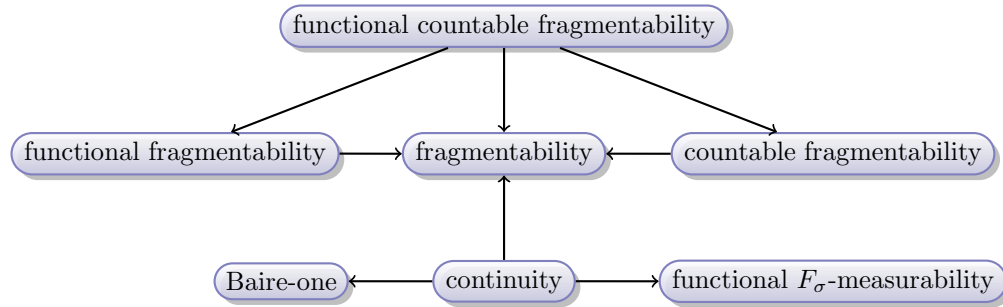
Proof. (1) \Rightarrow (2) is proved in [12, Proposition 3.1].

(2) \Rightarrow (1). We fix a nonempty closed set $F \subseteq X$. Denote $\beta = \min\{\xi \in [0, \alpha] : F \cap U_\xi \neq \emptyset\}$. Property (c) implies that $\beta = \xi + 1$ for some $\xi < \alpha$. Then the set $U = U_\beta \cap F$ is open in F and $\text{diam}f(U) \leq \text{diam}f(U_{\xi+1} \setminus U_\xi) < \varepsilon$. \square

If a sequence \mathcal{U} satisfies condition (2) of Proposition 1, then it is called ε -associated with f and is denoted by $\mathcal{U}_\varepsilon(f)$.

We say that an ε -fragmented map $f : X \rightarrow Y$ is *functionally ε -fragmented* if $\mathcal{U}_\varepsilon(f)$ can be chosen such that every set U_ξ is functionally open in X . Further, f is *functionally ε -countably fragmented* if $\mathcal{U}_\varepsilon(f)$ can be chosen to be countable and f is *functionally countably fragmented* if f is functionally ε -countably fragmented for all $\varepsilon > 0$.

Evident connections between kinds of fragmentability and its analogs are gathered in the following diagram.



Notice that none of the inverse implications is true.

Remark 1. (a) If X is hereditarily Baire, then every Baire-one map $f : X \rightarrow (Y, d)$ is barely continuous (i.e., for every nonempty closed set $F \subseteq X$ the restriction $f|_F$ has a point of continuity) and, hence, is fragmented (see [14, 31.X]).

(b) If X is a paracompact space in which every closed set is G_δ , then every fragmented map $f : X \rightarrow (Y, d)$ is Baire-one in the case either $\dim X = 0$, or Y is a metric contractible locally path-connected space [11, 12].

(c) Let $X = \mathbb{R}$ be endowed with the topology generated by the discrete metric $d(x, y) = 1$ if $x \neq y$, and $d(x, y) = 0$ if $x = y$. Then the identical map $f : X \rightarrow X$ is continuous, but is not countably fragmented.

For a deeper discussion of properties and applications of fragmented maps and their analogs we refer the reader to [1, 2, 7, 13, 15].

Proposition 2. *Let X be a topological space, (Y, d) be a metric space, $\varepsilon > 0$ and $f : X \rightarrow Y$ be a map. If one of the following conditions hold*

- (1) Y is separable and f is continuous,
- (2) X is metrizable separable and f is fragmented,
- (3) X is compact and $f \in B_1(X, Y)$,

then f is functionally countably fragmented.

Proof. Fix $\varepsilon > 0$.

(1) Choose a covering $(B_n : n \in \mathbb{N})$ of Y by open balls of diameters $< \varepsilon$. Let $U_0 = \emptyset$, $U_n = f^{-1}(\bigcup_{k \leq n} B_k)$ for every $n \in \mathbb{N}$ and $U_{\omega_0} = \bigcup_{n=0}^{\infty} U_n$. Then the sequence $(U_\xi : \xi \in [0, \omega_0])$ is ε -associated with f .

(2) Notice that any strictly increasing well-ordered chain of open sets in X is at most countable and every open set in X is functionally open.

(3) By [12, Proposition 7.1] there exist a metrizable compact space Z , a continuous function $\varphi : X \rightarrow Z$ and a function $g \in B_1(Z, \mathbb{R})$ such that $f = g \circ \varphi$. Then g is functionally ε -countably fragmented by condition (2) of the theorem. It is easy to see that f is functionally ε -countably fragmented too. \square

Lemma 3. *Let X be a topological space, $E \subseteq X$ and $f \in B_1(E, \mathbb{R})$. If there exists a sequence of functions $f_n \in B_1(X, \mathbb{R})$ such that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on E , then f can be extended to a function $g \in B_1(X, \mathbb{R})$.*

Proof. Without loss of generality we may assume that $f_0(x) = 0$ for all $x \in E$ and

$$|f_n(x) - f_{n-1}(x)| \leq \frac{1}{2^{n-1}}$$

for all $n \in \mathbb{N}$ and $x \in E$. Now we put

$$g_n(x) = \max\{\min\{(f_n(x) - f_{n-1}(x)), 2^{-n+1}\}, -2^{-n+1}\}$$

and notice that $g_n \in B_1(X, \mathbb{R})$. Moreover, the series $\sum_{n=1}^{\infty} g_n(x)$ is uniformly convergent on X for a function $g \in B_1(X, \mathbb{R})$. Then g is the required extension of f . \square

Recall that a subspace E of a topological space X is *z-embedded* in X if for any zero set F in E there exists a zero set H in X such that $H \cap E = F$; *C*-embedded* in X if any bounded continuous function f on E can be extended to a continuous function on X .

Proposition 4. *Let E be a z-embedded subspace of a completely regular space X and $f : E \rightarrow \mathbb{R}$ be a functionally countably fragmented function. Then f can be extended to a functionally countably fragmented function $g \in B_1(X, \mathbb{R})$.*

Proof. Let us observe that we may assume the space X to be compact. Indeed, E is z-embedded in βX , since X is C^* -embedded in βX [3, Theorem 3.6.1], and if we can extend f to a functionally countably fragmented function $h \in B_1(\beta X, \mathbb{R})$, then the restriction $g = h|_E$ is a functionally countably fragmented extension of f on X and $g \in B_1(X, \mathbb{R})$.

Fix $n \in \mathbb{N}$ and consider $\frac{1}{n}$ -associated with f sequence $\mathcal{U} = (U_\xi : \xi \leq \alpha)$. Without loss of the generality we can assume that all sets $U_{\xi+1} \setminus U_\xi$ are nonempty. Since E is z-embedded in X , one can choose a countable family $\mathcal{V} = (V_\xi : \xi \leq \alpha)$ of functionally open sets in X such that $V_\xi \subseteq V_\eta$ for all $\xi \leq \eta \leq \alpha$, $V_\xi \cap E = U_\xi$ for every $\xi \leq \alpha$ and $V_\eta = \bigcup_{\xi < \eta} V_\xi$ for every limit ordinal $\eta \leq \alpha$. For every $\xi \in [0, \alpha)$ we take an arbitrary point $y_\xi \in f(U_{\xi+1} \setminus U_\xi)$. Now for every $x \in X$ we put

$$f_n(x) = \begin{cases} y_\xi, & x \in V_{\xi+1} \setminus V_\xi, \\ y_0, & x \in X \setminus V_\alpha. \end{cases}$$

Observe that $f_n : X \rightarrow \mathbb{R}$ is functionally F_σ -measurable, since the preimage $f_n^{-1}(W)$ of any open set $W \subseteq \mathbb{R}$ is an at most countable union of functionally F_σ -sets from the system $\{V_{\xi+1} \setminus V_\xi : \xi \in [0, \alpha)\} \cup \{X \setminus V_\alpha\}$. Therefore, $f_n \in B_1(X, \mathbb{R})$.

It is easy to see that the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent to f on E . Now it follows from Lemma 3 that f can be extended to a function $g \in B_1(X, \mathbb{R})$. According to Proposition 2 (3), g is functionally countably fragmented. \square

Corollary 5. *Every functionally countably fragmented function $f : X \rightarrow \mathbb{R}$ defined on a topological space X belongs to the first Baire class.*

Proof. For every $n \in \mathbb{N}$ we choose a $\frac{1}{n}$ -associated with f family $\mathcal{U}_n = (U_{n,\xi} : \xi \leq \alpha_n)$ of functionally open sets $U_{n,\xi}$ and corresponding family $(\varphi_{n,\xi} : \xi \leq \alpha_n)$ of continuous functions $\varphi_{n,\xi} : X \rightarrow [0, 1]$ such that $U_{n,\xi} = \varphi_{n,\xi}^{-1}((0, 1])$. We consider the at most countable set $\Phi = \bigcup_{n=1}^{\infty} \{\varphi_{n,\xi} : 0 \leq \xi \leq \alpha_n\}$ and the continuous mapping $\pi : X \rightarrow [0, 1]^\Phi$, $\pi(x) = (\varphi(x))_{\phi \in \Phi}$.

Show that $f(x) = f(y)$ for every $x, y \in X$ with $\pi(x) = \pi(y)$. Let $x, y \in X$ with $\pi(x) = \pi(y)$. For every $n \in \mathbb{N}$ we choose $\xi_n \leq \alpha_n$ such that $x \in U_{n,\xi_n+1} \setminus U_{n,\xi_n}$. Then $y \in U_{n,\xi_n+1} \setminus U_{n,\xi_n}$ and

$$|f(x) - f(y)| \leq \text{diam}(U_{n,\xi_n+1} \setminus U_{n,\xi_n}) \leq \frac{1}{n}$$

for every $n \in \mathbb{N}$. Thus, $f(x) = f(y)$.

Now we consider the function $g : \pi(X) \rightarrow \mathbb{R}$, $g(\pi(x)) = f(x)$. Clearly, that every set $\pi(U_{n,\xi})$ is open in the metrizable space $\pi(X)$. Therefore, for every $n \in \mathbb{N}$ the family $(\pi(U_{n,\xi}) : \xi \leq \alpha_n)$ is $\frac{1}{n}$ -associated with g . Thus, g is functionally countably fragmented. According to Proposition 4, $g \in B_1(\pi(X), \mathbb{R})$. Therefore, $f \in B_1(X, \mathbb{R})$. \square

Combining Propositions 2 and 4 we obtain the following result.

Theorem 6. *Let X be a completely regular space. For a Baire-one function $f : X \rightarrow \mathbb{R}$ the following conditions are equivalent:*

- (1) f is functionally countably fragmented;
- (2) f can be extended to a Baire-one function on βX .

3. A Baire-one bounded function which is not countably fragmented

Theorem 7. *There exists a completely regular scattered (and hence hereditarily Baire) space X and a Baire-one function $f : X \rightarrow [0, 1]$ which can not be extended to a Baire-one function on βX .*

Proof. **Claim 1. Construction of X .** Let $\mathbb{Q} = \{r_n : n \in \mathbb{N}\}$, $r_n \neq r_m$ for all distinct $n, m \in \mathbb{N}$ and

$$\overline{\{r_{2n-1} : n \in \mathbb{N}\}} = \overline{\{r_{2n} : n \in \mathbb{N}\}} = \mathbb{Q},$$

$$A = \bigcup_{n=1}^{\infty} \{r_{2n-1}\} \times [0, 1], \quad B = \bigcup_{n=1}^{\infty} \{r_{2n}\} \times [0, 1].$$

We consider partitions $\mathcal{A} = (A_t : t \in [0, 1])$ and $\mathcal{B} = (B_t : t \in [0, 1])$ of the sets A and B into everywhere dense sets A_t and B_t , respectively, such that $|A_t| = |B_t| = \mathfrak{c}$. Moreover, let $[0, 1] = \bigsqcup_{\alpha < \omega_1} T_\alpha$ with $|T_\alpha| = \mathfrak{c}$ for every $\alpha < \omega_1$. For every $\alpha \in [0, \omega_1)$ we put

$$Q_\alpha = \begin{cases} \bigsqcup_{t \in T_\alpha} A_t, & \alpha \text{ is even,} \\ \bigsqcup_{t \in T_\alpha} B_t, & \alpha \text{ is odd,} \end{cases}$$

$$Q = \bigsqcup_{\alpha < \omega_1} Q_\alpha, \quad X_\alpha = Q_\alpha \times \{\alpha\} \quad \text{and} \quad X = \bigsqcup_{\alpha < \omega_1} X_\alpha.$$

Claim 2. Indexing of X . For every $\alpha \in [0, \omega_1)$ we consider the set

$$I_\alpha = \{(i_\xi)_{\xi \in [\alpha, \omega_1)} : |\{\xi : i_\xi \neq 0\}| \leq \aleph_0\} \subseteq [0, 1]^{[\alpha, \omega_1)}$$

and notice that $|I_\alpha| = \mathfrak{c}$. Let $\varphi_\alpha : I_\alpha \rightarrow T_\alpha$ be a bijection and

$$X_\alpha = \bigsqcup_{j \in I_{\alpha+1}} X_{\alpha,j},$$

where

$$X_{\alpha,j} = \begin{cases} A_{\varphi_{\alpha+1}(j)} \times \{\alpha\}, & \alpha \text{ is even,} \\ B_{\varphi_{\alpha+1}(j)} \times \{\alpha\}, & \alpha \text{ is odd,} \end{cases}$$

For all $\xi, \eta \in [0, \omega_1)$ with $\eta > \xi$ and $i \in I_\eta$ we put

$$J_{\eta,\xi}^i = \{j \in I_\xi : j|_{[\eta, \omega_1)} = i\}.$$

In particular, if $\xi = \alpha$, $\eta = \alpha + 1$ and $i \in I_{\alpha+1}$, then we denote the set $J_{\alpha+1,\alpha}^i$ simply by J_α^i . Notice that $|J_\alpha^i| = \mathfrak{c}$ and we may assume that

$$X_{\alpha,i} = \{x_j : j \in J_\alpha^i\}.$$

Then

$$X_\alpha = \{x_i : i \in I_\alpha\},$$

since $I_\alpha = \bigsqcup_{i \in I_{\alpha+1}} J_\alpha^i$.

Claim 3. Topologization of X . For all $\alpha \in [1, \omega_1)$, $i \in I_\alpha$ and $x = x_i \in X_\alpha$ we put

$$L_{<x} = \bigsqcup_{\xi < \alpha} \{x_j \in X_\xi : j \in J_{\alpha,\xi}^i\}, \quad L_{\leq x} = \{x\} \cup L_{<x}.$$

Notice that for all $x \in X_\alpha$ and $y \in X_\beta$ with $\alpha \leq \beta$ either $L_{\leq x} \subseteq L_{\leq y}$, or $L_{\leq x} \cap L_{\leq y} = \emptyset$.

Now we are ready to define a topology τ on X . Each point of X_0 is isolated. For any $\alpha \in [1, \omega_1)$ and a point $x \in X_\alpha$ we construct a base \mathcal{U}_x of τ -open neighborhoods of x in the following way. Take $i \in I_\alpha$ and $q \in Q$ such that $x = x_i = (q, \alpha) \in X_\alpha$. Let \mathcal{V}_q be a base of clopen neighborhoods of q in the space Q equipped with the topology induced from \mathbb{R}^2 . Then we put

$$\mathcal{U}_x = \{(V \times [0, \omega_1)) \cap (L_{\leq x} \setminus \bigcup_{y \in Y} L_{\leq y}) : V \in \mathcal{V}_q \text{ and } Y \subseteq L_{< x} \text{ is finite}\}.$$

Claim 4. Complete regularity and scatteredness of X . We show that the space (X, τ) is completely regular. We prove firstly that every set $L_{\leq x}$ is clopen. Since the inclusion $v \in L_{\leq u}$ implies $L_{\leq v} \subseteq L_{\leq u}$, every set $L_{\leq x}$ is open. Now let $y \in \overline{L_{\leq x}}$. Then $L_{\leq y} \cap L_{\leq x} \neq \emptyset$. Therefore, $L_{\leq y} \subseteq L_{\leq x}$ or $L_{\leq x} \subseteq L_{\leq y}$. Assume that $y \notin L_{\leq x}$. Then $x \in L_{< y}$ and for a neighborhood $W = L_{\leq y} \setminus L_{\leq x}$ of y we have $W \cap L_{\leq x} = \emptyset$, which implies a contradiction. Thus, $\overline{L_{\leq x}} = L_{\leq x}$ and the set $L_{\leq x}$ is closed.

Notice that for every clopen in Q set V the set $(V \times [0, \omega_1)) \cap X$ is clopen in X . Therefore, every $U \in \mathcal{U}_x$ is clopen in X for every $x \in X$. In particular, (X, τ) is completely regular.

In order to show that (X, τ) is scattered we take an arbitrary nonempty set $E \subseteq X$ and denote $\alpha = \min\{\xi \in [0, \omega_1) : E \cap X_\xi \neq \emptyset\}$. Then any point x from $E \cap X_\alpha$ is isolated in E .

Claim 5. $\overline{X_\alpha} = \bigcup_{\xi \geq \alpha} X_\xi$ for every $\alpha \in [0, \omega_1)$. It is sufficient to prove that

$$X_\alpha \subseteq \overline{\bigcup_{\xi < \alpha} X_\xi}$$

for every $\alpha \in [1, \omega_1)$. Let $\alpha \in [1, \omega_1)$, $x = (q, \alpha) \in X_\alpha$, V be an open neighborhood of q in Q , $Y \subseteq L_{< x}$ be a finite set and

$$U = (V \times [0, \omega_1)) \cap (L_{\leq x} \setminus \bigcup_{y \in Y} L_{\leq y}).$$

We show that $U \cap (\bigcup_{\xi < \alpha} X_\xi) \neq \emptyset$. For every $y \in Y$ we choose $\beta_y < \alpha$ such that $y \in X_{\beta_y}$. We put $\beta = \max\{\beta_y : y \in Y\}$ and $\gamma = \beta + 1$. Since $\beta < \alpha$, $\gamma \leq \alpha$. We choose $i \in I_\alpha$ such that $x = x_i$ and choose $j \in J_\gamma$ such that $j|_{[\alpha, \omega_1)} = i$. We consider the set $X_{\beta, j}$. Recall that $X_{\beta, j} = A_t \times \{\beta\}$ or $X_{\beta, j} = B_t \times \{\beta\}$, where $t = \varphi_\gamma(j)$. Therefore, the set

$$P = \{p \in Q : (p, \beta) \in X_{\beta, j}\}$$

is dense in Q . Thus, the set $P \cap V$ is infinite. Moreover, $|L_{\leq y} \cap X_\beta| \leq 1$ for every $y \in Y$. Hence, the set

$$S = \{p \in P : (p, \beta) \in \bigcup_{y \in Y} L_{\leq y}\}$$

is finite. Therefore, the set $(P \cap V) \setminus S$ is infinite, in particular, it is nonempty. We choose a point $p \in (P \cap V) \setminus S$. Then $z = (p, \beta) \in X_{\beta, j}$. Thus, $z \in L_{\leq x}$. Moreover, $z \in V \times [0, \omega_1)$ and $z \notin \bigcup_{y \in Y} L_{\leq y}$. Thus, $z \in U$. Since $X_{\beta, j} \subseteq X_\beta$, $z \in \bigcup_{\xi < \alpha} X_\xi$. Therefore, $U \cap \bigcup_{\xi < \alpha} X_\xi \neq \emptyset$ for every $U \in \mathcal{U}_x$ and $x \in \overline{\bigcup_{\xi < \alpha} X_\xi}$.

Claim 6. Construction of a Baire-one function f . We put

$$C = \bigsqcup_{\xi < \omega_1, \xi \text{ is even}} X_\alpha$$

and show that the function $f : X \rightarrow [0, 1]$,

$$f = \chi_C,$$

belongs to the first Baire class.

Consider a mapping $\pi : X \rightarrow \mathbb{Q}$, $\pi(x) = r$ if $x = (q, \alpha)$ and $q = (r, t)$ for some $t \in [0, 1]$ and $\alpha < \omega_1$. The mapping π is continuous, because for every open in \mathbb{Q} set V the set

$$\pi^{-1}(V) = (V \times [0, 1] \times [0, \omega_1)) \cap X$$

is open in X . Clearly, the function $g : \mathbb{Q} \rightarrow \mathbb{R}$,

$$g(t) = \begin{cases} 1, & t = r_{2n-1}, \\ 0, & t = r_{2n}, \end{cases}$$

belongs to the first Baire class. Therefore, the function $f(x) = g(\pi(x))$ belongs to the first Baire class too.

Claim 7. The function f is not countably fragmented. Finally, we prove that f is not countably fragmented. Assume the contrary and take a countable sequence $\mathcal{U} = (U_\xi : \xi < \alpha)$ of functionally open sets such that \mathcal{U} is $\frac{1}{2}$ -associated with f .

We show that $U_\beta \subseteq \bigcup_{\xi < \beta} X_\xi$ for every $\beta \leq \alpha$. We will argue by induction on β . For $\beta = 0$ the assertion is obvious. Assume that the inclusion is valid for all $\beta < \gamma \leq \alpha$. If γ is a limit ordinal, then

$$U_\gamma = \bigcup_{\xi < \gamma} U_\xi \subseteq \bigcup_{\xi < \gamma} \bigcup_{\eta < \xi} X_\eta = \bigcup_{\xi < \gamma} X_\xi.$$

Now let $\gamma = \delta + 1$. Suppose that there exists $x \in U_\gamma \setminus (\bigcup_{\xi \leq \delta} X_\xi)$. Notice that $x \in \overline{X_\gamma} \subseteq \overline{X_\delta}$ according to Claim 6. Therefore, there exist $z_1 \in U_\gamma \cap X_\gamma$ and $z_2 \in U_\gamma \cap X_\delta$. According to the inductive assumption, we have $U_\delta \subseteq \bigcup_{\xi < \delta} X_\xi$. Therefore, $z_1, z_2 \subseteq U_\gamma \setminus U_\delta = U_{\delta+1} \setminus U_\delta$. Thus, we have

$$1 = |f(z_1) - f(z_2)| \leq \text{diam}(U_{\delta+1} \setminus U_\delta) < \frac{1}{2},$$

a contradiction.

Theorem 6 implies that f can not be extended to a Baire-one function on βX . □

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